JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 144, 236–241 (1989)

A Question of Hille on Inversion of Differential Operators of Infinite Order

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Submitted by R. P. Boas

Received December 16, 1987

The inversion problem for differential equations of the form $G(D_z) = F(z)$ is studied. Here D_z is a linear differential operator of order n having entire coefficients with the leading coefficient a polynomial whose zeros are suitably restricted. The function G is entire of order at most $1/n$, minimal type, and the function F is analytic in a neighborhood of an interval on the real axis. Conditions for obtaining an analytic solution W of the above equation are given. © 1989 Academic Press, Inc.

INTRODUCTION

The purpose of the current note is to answer a question posed by Hille in a paper [2] on the solutions of differential equations of the form

$$G(D_z)\,W(z) = F(z), \tag{1}$$

where

$$D_z = \sum_{j=0}^{n} P_{n-j}(z)\,W^{(j)}(z) \tag{2}$$

for $n \geqslant 2$. The P_{n-j} are entire functions, real valued on the real axis, with $P_0 \not\equiv 0$ a polynomial whose zeros are suitably restricted. The function $G(w)$ is entire of order at most $1/n$ and is of minimal type if the order is $1/n$. Moreover, we shall assume that G has only positive zeros in the interval $[A, \infty)$, with A to be specified later. Let $F(z)$ be analytic on a domain Δ which contains the interval (a, b) on which the eigenvalue problem (3) is defined. The problem is to solve Eq. (1) for $W(z)$ such that $W(z)$ is also analytic in Δ. This amounts to inverting the operator $G(D_z)$. To further explain the setting, we make use of various facts presented by Hille.

0022-247X/89 $3.00

To start, let (a, b) be an interval on the real axis which is free of zeros of $P_0(z)$. On (a, b) we pose the eigenvalue problem

$$D_z W(z, \lambda) = \lambda W(z, \lambda). \tag{3}$$

Let $W_j(x, \lambda)$, for $x \in (a, b)$, denote the fundamental solutions of (3) satisfying the initial conditions

$$W_j^{(k)}(x_0, \lambda) = \delta_{jk},$$

where $x_0 \in (a, b)$, $1 \leqslant j \leqslant n$, $0 \leqslant k \leqslant (n-1)$. The functions $W_j(x, \lambda)$ are entire functions of λ of order $1/n$, mean type. The boundary conditions for the eigenvalue problem (3) may be either regular or singular and (a, b) may be either finite or infinite. In particular, if a, b are both finite and not singular points, then regular boundary conditions are used. If one of the finite endpoints is singular, assume that

$$W(x, \lambda), ..., W^{(n-1)}(x, \lambda) \in L_2(a, b).$$

If a or b is infinite and $P_0(z)$ is not constant, then assume that

$$W(x, \lambda)\, |P_0(x)|^{-1/2}, ..., W^{(n-1)}(x, \lambda)\, |P_0(x)|^{-1/2} \in L_2(a, b).$$

Now, set

$$W_{kj}(z, \lambda) = W_j^{(k)}(z, \lambda)$$

and form the matrix differential equation

$$(d/dz)\, W(z, \lambda) = A(z)\, W(z, \lambda), \tag{4}$$

where

$$A(z) = \begin{bmatrix} 0 \quad 1 \quad 0 & \cdots & 0 \\ 0 \quad 0 \quad 1 \quad 0 & \cdots & 0 \\ 0 \quad \cdots & & 1 \\ -P_n(z) + \lambda - P_{n-1}(z) & & -P_1(z) \end{bmatrix}$$

and

$$W(z, \lambda) = \begin{bmatrix} W_{1,0}(z, \lambda) & \cdots & W_{n,0}(z, \lambda) \\ \vdots & & \vdots \\ W_{1,n-1}(z, \lambda) & \cdots & W_{n,n-1}(z, \lambda) \end{bmatrix}.$$

Next, let $F(t) \not\equiv 0$ be a function of bounded variation on $[a, b]$ and form

$$H(\lambda) = \int_a^b W(t, \lambda)\, dF(t),$$

and let

$$h(\lambda) = \det H(\lambda). \tag{5}$$

Now, $h(\lambda)$ is entire having order at most $1/n$ and mean type. Consider the regular boundary value problem

$$(d/dz)\, W(z, \lambda) = A(z)\, W(z, \lambda),\ h(\lambda) = 0. \tag{6}$$

The eigenvalues λ_n for Problem (6) are the zeros of $h(\lambda)$ and the eigenfunctions are the matrices $W(z, \lambda_n)$. Moreover, the entries $W_j(z, \lambda_n)$ of the first row of $W(z, \lambda)$ are solutions to (3). As Hille does, we make the following assumptions:

I. All but a finite number of the eigenvalues λ_n are real negative.

II. The eigenfunctions $\omega_j(x)$ of D_z are an orthogonal system, complete in $L_2(a, b)$.

III. There exists a bounded continuous function U on (a, b) for which

$$|W_j(x, \lambda_n)| \leqslant |\lambda_n|^{\alpha_{1,j}}\, U(x), \ldots, |W_j^{(n-1)}(x, \lambda_n)| \leqslant |\lambda_n|^{\alpha_{n-1,j}}\, U(x)$$

for $1 \leqslant j \leqslant n$.

The eigenfunctions $\omega_j(z)$ for the operator D_z are analytic functions of z in some domain Δ containing the finite interval $[a, b]$. Next, let Δ_1 be a finite domain containing $[a, b]$ and contained in Δ and let $F(\Delta_1)$ be the class of all functions analytic on Δ_1.

Hille proved the following result in [2]:

THEOREM A. *Let $\{\lambda_n, \omega_j(x)\}$ be the regular eigensystem for D_z on $[a, b]$. Let Δ_1 be a bounded domain containing $[a, b]$ and none of the zeros of $P_0(z)$. If (1) has a solution $W(z)$ in $F(\Delta_1)$ for every $F(z)$ in $F(\Delta_1)$, then there must exist positive numbers B and M such that*

$$B^{-1} \leqslant |G(\lambda_n)| \leqslant M$$

for all n.

In the case that $\{\lambda_n, \omega_j\}$ is a singular eigensystem for D_z, e.g., if $(a, b) = (-\infty, \infty)$, as Hille points out, Theorem A remains true provided the class $F(\Delta_1)$ is restricted to those functions whose Fourier coefficients exist as in (7) and (8), given below.

Following Hille, we see that if (1) has a solution in $F(\Delta_1)$, then we can form a Fourier expansion for W which is formally given

$$W(x) \sim \sum_{j=0}^{\infty} W_{0j}\omega_j(x). \tag{7}$$

The function $F(x)$ has the expansion

$$F(x) \sim \sum_{j=0}^{\infty} F_j\omega_j(x), \tag{8}$$

where F_j and $W_{0,j}$ are related by

$$G(\lambda_j)\ W_{0,j} = F_j. \tag{9}$$

We can therefore formally express $W(x)$ as

$$W(x) \sim \sum_{j=0}^{\infty} F_j[G(\lambda_j)^{-1}]\ \omega_j(x), \tag{10}$$

which lies in $L_2(a, b)$ if and only if

$$\sum_{j=0}^{\infty} (|F_j[G(\lambda_j)]^{-1}|)^2 < \infty.$$

We assume that the positive zeros of G are larger than the largest real eigenvalue given in Assumption I. As G is entire of order at most $1/n$, minimal type, by surrounding the zeros $\{w_k\}$ of $G(z)$ by disks

$$D_k = \{w\colon |w - w_k| \leqslant |w_k|^{-2/n}\},$$

Hille [3, Chap. 14] proves that in the exterior of these disks $G(w)$ satisfies

$$|G(w)| > \exp(-|w|^{1/n}). \tag{11}$$

Hille concludes that for the Fourier expansion in (10) to converge in $L_2(a, b)$, two results suffice. One is that the spectrum $\{\lambda_j\}$ lie outside the disks D_k. The other is that the series

$$\sum_{j=0}^{\infty} |F_j \exp(|\lambda_j|^{1/n})|^2 < \infty. \tag{12}$$

Hille's question is: under what conditions is the solution $W(x)$ to (1) extendable by analytic continuation to a function analytic in Δ_1?

THEOREM

We now answer Hille's question.

THEOREM. *Assuming that the eigenfunctions ω_j in* (7) *and* (8) *are analytic functions in Δ_1, the condition*

$$\limsup_{j\to\infty} |F_j \exp(\tau\, |\lambda_j|^{(1/n)+\varepsilon})| < \infty \tag{13}$$

for every $\tau > 0$ and every $\varepsilon > 0$ is sufficient for $W(x)$ to extend by analytic continuation to a function analytic in the domain Δ_1 and is a solution to (1). *Moreover, the condition*

$$\sum_{j=0}^{\infty} |F_j \exp(\tau\, |\lambda_j|^{(1/n)+\varepsilon})|^2 < \infty \tag{14}$$

for every $\tau > 0$ and every ε, $0 < \varepsilon < 1/n$, is also sufficient.

Proof. The eigenfunctions $\omega_j(z)$ can be expressed in terms of the fundamental solutions $W_1(z, \lambda_j)$, ..., $W_{n-1}(z, \lambda_j)$ by means of

$$\omega_j(z) = \omega_j(x_0)\, W_1(z, \lambda_j) + \cdots + \omega_j^{(n-1)}(x_0)\, W_{n-1}(z, \lambda_j). \tag{15}$$

Now, let K be a compact subset of Δ_1. For all $z \in K$, for every $\varepsilon > 0$, and for each $l = 1, ..., n$, there exist constants M_l and N_l such that

$$|W_l(z, \lambda)| \leqslant M_l \exp(N_l\, |\lambda|^{(1/n)+\varepsilon}) \leqslant M \exp(N\, |\lambda|^{(1/n)+\varepsilon})$$

where

$$M = \max\{M_l : 1 \leqslant l \leqslant n\},$$
$$N = \max\{N_l : 1 \leqslant l \leqslant n\}.$$

We obtain from (15) and Assumption III that

$$|\omega_j(z)| \leqslant CM[|\lambda_j|^{\alpha_1} + \cdots + |\lambda_j|^{\alpha_{n-1}}] \exp[N\, |\lambda_j|^{(1/n)+\varepsilon}]$$

for some constants C and $\alpha_1 = \alpha_{1,j}$, ..., $\alpha_n = \alpha_{n,j}$. As the eigenvalues λ_j are zeros of the function $h(\lambda)$ in (5), which has order $1/n$, mean type, by a standard fact in entire function theory [1], the exponent of convergence of the λ_j is at most $1/n$. Consequently, for j sufficiently large, there exist constants $\tilde{M}$, $\tilde{N}$ such that

$$|\omega_j(z)| \leqslant \tilde{M}\, |\lambda_j|^{\alpha_1 + \cdots + \alpha_{n-1}} \exp[\tilde{N}\, |\lambda_j|^{(1/n)+\varepsilon}].$$

Now, by (13), choose $\mu > \tilde{N}$ such that for some constant P

$$|F_j| \leqslant P \exp[(\tilde{N} - \mu)\, |\lambda_j|^{(1/n)+\varepsilon}].$$

In addition, for j sufficiently large, we have that

$$\exp(-\mu)\, |\lambda_j|^{(1/n)+\varepsilon}) \leqslant |\lambda_j|^{-(\alpha_1 + \cdots + \alpha_{n-1} + \sigma)},$$

where $\sigma > 1/n$. Therefore the series

$$\sum_{j=0}^{\infty} F_j \omega_j(z)$$

has the property that there exist constants D and J such that

$$\sum_{j \geqslant J} |F_j \omega_j(z)| \leqslant D \sum_{j \geqslant J} |\lambda_j|^{-\sigma},$$

which converges. Therefore, using (11),

$$W(x) = \sum_{j=0}^{\infty} F_j [G(\lambda_j)]^{-1}\, \omega_j(x)$$

admits analytic continuation into $K \subset \Delta_1$. Hence, Condition (12) suffices to solve the problem.

Finally, the condition

$$\sum_{j=0}^{\infty} |F_j \exp(\tau\, |\lambda_j|^{(1/n)+\varepsilon})|^2 < \infty$$

implies that for given $\eta > 0$ and for j sufficiently large

$$|F_j| \leqslant \eta \exp(-\tau\, |\lambda_j|^{(1/n)+\varepsilon})$$

for every ε, $0 < \varepsilon < 1/n$. The above argument may now be applied.

In his paper [2], Hille states other interesting open questions regarding the convexity and connectivity of the domains of absolute convergence for eigenfunction expansions. These questions warrant further investigation.

ACKNOWLEDGMENT

The author is partially supported by NSF Grant RII 8410705.

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